

Minimum Lap Time Trajectory Optimization for an F1 Track

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1 Headlines

Task Assignment:

- **Introduction:** Valerie, Julie, Shaurya, Samar
- **Problem Statement:** All
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- **KKT Conditions:** Valerie, Julie, Shaurya, Samar
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- **Conclusion:** Valerie, Julie, Shaurya

2 Introduction

2.1 Motivation

While professional racing teams rely on computational fluid dynamics simulations and iterative track testing to develop good racing lines, a rigorous optimization framework can reveal fundamental insights about the structure of vehicle control and the interplay between path geometry, velocity profiles, and traction management. The problem has mostly convex structures that make it particularly amenable to sequential convex programming approaches, enabling potential real-time applications in autonomous racing and driver assistance systems. In addition, a friction circle constraint introduces elegant coupling between physics trajectory planning and speed profile optimization that mirrors classical problems in robotics and aerospace.

2.2 Previous Works

- van den Eshof et al. (2025): “A Computationally Efficient Framework for Free-trajectory Minimum Lap-Time Optimization” — Sequential convex second-order conic programming achieving convergence in seconds
<https://arxiv.org/html/2511.13522v1>
- Parwana et al. (2022): “Recursive Feasibility Guided Optimal Parameter Adaptation of Differential Convex Model Predictive Control” — Approaches for maintaining feasibility in sequential optimization
https://websites.umich.edu/~dpanagou/assets/documents/HParwana_ICRA22.pdf

2.3 Intended Contributions

1. Friction circle constraint: Develop a sequential convex programming framework that handles the nonconvex coupling between downforce and available tire grip. Higher speed $\rightarrow$ more downforce $\rightarrow$ larger friction circle $\rightarrow$ can corner harder AND accelerate harder simultaneously.

2. Optimize position and speed: Formulate the free-trajectory problem where lateral track position and speed are optimized together, showing lap time improvements over fixed “minimum-curvature” baselines. We will compare sequential linearization (SL) vs sequential convex restriction (SCR) approaches for maintaining feasibility. In our case, we have s , which serves as our progress along the centerline, and all of our physical constraints are in the perspective of the centerline. In this way, we are formulating our problem as a Frenet-Serret frame, where our coordinate system is moving along the curve of a centerline track. Therefore, our optimal line and centerline do not need to be convex from the perspective of Cartesian coordinate systems.

3. Testing: Compare optimized trajectories with telemetry from actual F1 qualifying laps to assess model fidelity and identify the impact of simplified effects like load-dependent grip coefficients and cornering-induced resistances.

2.4 Organization of the Paper

The rest of the paper is organized as follows. Section III introduces the problem formulation where the primal and dual formulations are presented along with the KKT conditions. Section IV outlines our solution approach, namely our sequential convex programming framework that addresses the convexity of the trajectory optimization problem. Section V presents the experimental results and evaluates the performance of our solution. Lastly, Section VI concludes by talking about key findings, limitations, and possible directions for future work.

3 Statement of the Problem

3.1 Primal Formulation

State and Control Variables

States $\in \mathbb{R}^3$:

- n_i : Lateral offset from track centerline [m]
- α_i : Heading angle relative to centerline [rad]
- E_i : Kinetic energy, $E_i = \frac{1}{2}mv_i^2$ [J]

Controls $\in \mathbb{R}^2$:

- $F_{x,i}$: Longitudinal tire force [N]
- $F_{y,i}$: Lateral tire force [N]

Auxiliary Variables $\in \mathbb{R}^2$:

- p_i : Pace (inverse squared velocity measure) [s/m]
- $\kappa_{\text{path},i}$: Path curvature [1/m]

$$t_{\text{lap}} = \min_{\{n_i, \alpha_i, E_i, p_i, \kappa_{\text{path}, i}, F_{x, i}, F_{y, i}\}} \sum_{i=0}^{N-1} p_i \cdot \Delta s$$

subject to:

1. Track Boundary Constraints (Linear)

$$n_{\min}(s_i) + \frac{w_{\text{veh}}}{2} \leq n_i \leq n_{\max}(s_i) - \frac{w_{\text{veh}}}{2}, \quad i = 0, \dots, N \quad (1)$$

2. Longitudinal Dynamics (Linear)

$$\frac{E_{i+1} - E_i}{\Delta s} = F_{x, i} - \left(\frac{2C_d}{m} \right) E_i, \quad i = 0, \dots, N-1 \quad (2)$$

3. Kinematic Constraints (Bilinear)

$$\frac{n_{i+1} - n_i}{\Delta s} = \alpha_i, \quad i = 0, \dots, N-1 \quad (3)$$

$$\frac{\alpha_{i+1} - \alpha_i}{\Delta s} = \kappa_{\text{path}, i} - \kappa_{\text{ref}, i}, \quad i = 0, \dots, N-1 \quad (4)$$

$$F_{y, i} = 2\kappa_{\text{path}, i} E_i, \quad i = 0, \dots, N \quad (5)$$

4. Friction Circle (SOCP)

$$\left\| \begin{pmatrix} F_{x, i} \\ F_{y, i} \end{pmatrix} \right\|_2 \leq \mu m g, \quad i = 0, \dots, N \quad (6)$$

5. Velocity–Pace Coupling (Rotated SOCP)

$$\left\| \begin{pmatrix} \sqrt{2m} \\ p_i - E_i \end{pmatrix} \right\|_2 \leq p_i + E_i, \quad i = 0, \dots, N \quad (7)$$

6. Vehicle Box Constraints (Linear)

$$0 \leq E_i \leq E_{\max}, \quad i = 0, \dots, N \quad (8)$$

$$-F_{\text{brake}, \max} \leq F_{x, i} \leq F_{\text{accel}, \max}, \quad i = 0, \dots, N \quad (9)$$

$$|\alpha_i| \leq \alpha_{\max}, \quad i = 0, \dots, N \quad (10)$$

7. Periodicity (Closed Lap)

$$n_0 = n_N, \quad \alpha_0 = \alpha_N, \quad E_0 = E_N \quad (11)$$

Parameters

Parameter	Description
m	Vehicle mass [kg]
g	Gravitational acceleration [m/s ²]
μ_x, μ_y	Longitudinal and lateral friction coefficients
w_{veh}	Vehicle width [m]
Δs	Spatial discretization step [m]
N	Number of discretization points
κ_{path}	The curvature of the driver's line
κ_{ref}	The curvature of the center line
x	Horizontal position of the car
y	Vertical position of the car

Note on Velocity-Lethargy Coupling

Due to the use of kinetic energy E_i as a state, our pace/lethargy p_i is the inverse of E_i :

$$\begin{aligned}
 p_i &= \frac{1}{v_i^2} \\
 E_i &= \frac{1}{2}mv^2 \\
 E_i &= \frac{1}{2}m\left(\frac{1}{p_i}\right) \\
 2p_iE &= m
 \end{aligned}$$

This becomes an inequality constraint where we attempt to minimize p_i from our primal until our inequality becomes tight.

$$\begin{aligned}
 2p_iE &\geq m \\
 4p_iE &\geq 2m \\
 4p_iE &\geq 2m \\
 (p_i + E)^2 - (p_i - E)^2 &\geq 2m \\
 (p_i + E)^2 &\geq 2m + (p_i - E)^2 \\
 (p_i + E)^2 &\geq (\sqrt{2m})^2 + (p_i - E)^2 \\
 p_i + E &\geq \sqrt{(\sqrt{2m})^2 + (p_i - E)^2} \\
 \left\| \begin{pmatrix} \sqrt{2m} \\ p_i - E_i \end{pmatrix} \right\|_2 &\leq p_i + E_i
 \end{aligned}$$

3.2 Dual Formulation

Lagrangian Multipliers

Track Bounds: $\underline{\lambda}_i^n \geq 0, \bar{\lambda}_i^n \geq 0$

Longitudinal Dynamics: $\nu_i^E \in \mathbb{R}$

Kinematics: $\nu_i^n, \nu_i^\alpha \in \mathbb{R}$ and $\nu_i^F \in \mathbb{R}$
 Box Constraints: $\underline{\lambda}_i^E, \bar{\lambda}_i^E, \underline{\lambda}_i^{Fx}, \bar{\lambda}_i^{Fx}, \underline{\lambda}_i^\alpha, \bar{\lambda}_i^\alpha \geq 0$
 Periodicity: $\gamma^n, \gamma^\alpha, \gamma^E \in \mathbb{R}$
 Friction Circle: $z_i^{\text{fric}} = (u_{1,i}, v_{1x,i}, v_{1y,i}) \in \mathbb{R}^3$
 Velocity-Pace: $z_i^{\text{pace}} = (u_{2,i}, v_{2a,i}, v_{2b,i}) \in \mathbb{R}^3$

Lagrangian

$$\begin{aligned}
 \mathcal{L}_{x,\lambda,\nu,z} = & \sum_{i=0}^{N-1} p_i \Delta s \\
 & + \sum_{i=0}^N \underline{\lambda}_i^n \left(n_{\min}(s_i) + \frac{w_{\text{veh}}}{2} - n_i \right) + \sum_{i=0}^N \bar{\lambda}_i^n \left(n_i - n_{\max}(s_i) + \frac{w_{\text{veh}}}{2} \right) \\
 & + \sum_{i=0}^{N-1} \nu_i^E \left(E_{i+1} - E_i - \Delta s F_{x,i} + \Delta s \frac{2C_d}{m} E_i \right) \\
 & + \sum_{i=0}^{N-1} \nu_i^n (n_{i+1} - n_i - \Delta s \alpha_i) \\
 & + \sum_{i=0}^{N-1} \nu_i^\alpha (\alpha_{i+1} - \alpha_i - \Delta s (\kappa_{\text{path},i} - \kappa_{\text{ref},i})) \\
 & + \sum_{i=0}^N \nu_i^F (F_{y,i} - 2\kappa_{\text{path},i} E_i) \\
 & - \sum_{i=0}^N (u_{1,i} \mu m g + v_{1x,i} F_{x,i} + v_{1y,i} F_{y,i}) \\
 & - \sum_{i=0}^N \left(u_{2,i} (p_i + E_i) + v_{2a,i} \sqrt{2m} + v_{2b,i} (p_i - E_i) \right) \\
 & + \sum_{i=0}^N \left[\underline{\lambda}_i^E (-E_i) + \bar{\lambda}_i^E (E_i - E_{\max}) + \underline{\lambda}_i^{Fx} (-F_{\text{brake,max}} - F_{x,i}) \right. \\
 & \quad \left. + \bar{\lambda}_i^{Fx} (F_{x,i} - F_{\text{accel,max}}) + \underline{\lambda}_i^\alpha (-\alpha_{\max} - \alpha_i) + \bar{\lambda}_i^\alpha (\alpha_i - \alpha_{\max}) \right] \\
 & + \gamma^n (n_0 - n_N) + \gamma^\alpha (\alpha_0 - \alpha_N) + \gamma^E (E_0 - E_N)
 \end{aligned}$$

Dual Function

$$\begin{aligned}
g(\lambda, \nu, z) &= \inf_{x \in \mathcal{X}} \mathcal{L}(x, \lambda, \nu, z) \\
&= \sum_{i=0}^N \left[\underline{\lambda}_i^n \left(n_{\min}(s_i) + \frac{w_{\text{veh}}}{2} \right) - \bar{\lambda}_i^n \left(n_{\max}(s_i) - \frac{w_{\text{veh}}}{2} \right) \right] \\
&\quad + \sum_{i=0}^{N-1} \nu_i^\alpha \Delta s \kappa_{\text{ref},i} \\
&\quad - \sum_{i=0}^N u_{1,i} \mu m g - \sum_{i=0}^N v_{2a,i} \sqrt{2m} \\
&\quad - \sum_{i=0}^N \left[\bar{\lambda}_i^E E_{\max} + \bar{\lambda}_i^{Fx} F_{\text{accel},\max} + \underline{\lambda}_i^{Fx} F_{\text{brake},\max} + (\bar{\lambda}_i^\alpha + \underline{\lambda}_i^\alpha) \alpha_{\max} \right]
\end{aligned}$$

Final Dual Conditions

- With respect to p_i :

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial p_i} &= \frac{\partial}{\partial p_i} \left(\sum_{i=0}^{N-1} p_i \Delta s - \sum_{i=0}^N \left(u_{2,i}(p_i + E_i) + v_{2a,i} \sqrt{2m} + v_{2b,i}(p_i - E_i) \right) \right) \\
&= \Delta s - u_{2,i} - v_{2b,i} = 0
\end{aligned}$$

- With respect to $F_{x,i}$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial F_{x,i}} &= \frac{\partial}{\partial F_{x,i}} \left(\sum_{i=0}^{N-1} \nu_i^E \left(E_{i+1} - E_i - \Delta s F_{x,i} + \Delta s \frac{2C_d}{m} E_i \right) \right. \\
&\quad - \sum_{i=0}^N (u_{1,i} \mu m g + v_{1x,i} F_{x,i} + v_{1y,i} F_{y,i}) \\
&\quad + \sum_{i=0}^N [\underline{\lambda}_i^E (-E_i) + \bar{\lambda}_i^E (E_i - E_{\max}) + \underline{\lambda}_i^{Fx} (-F_{\text{brake},\max} - F_{x,i}) \\
&\quad \quad \left. + \bar{\lambda}_i^{Fx} (F_{x,i} - F_{\text{accel},\max}) + \underline{\lambda}_i^\alpha (-\alpha_{\max} - \alpha_i) + \bar{\lambda}_i^\alpha (\alpha_i - \alpha_{\max}) \right] \Big) \\
&= -\nu_i^E \Delta s - v_{1x,i} - \underline{\lambda}_i^{Fx} + \bar{\lambda}_i^{Fx} = 0
\end{aligned}$$

- With respect to $F_{y,i}$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial F_{y,i}} &= \frac{\partial}{\partial F_{y,i}} \left[\sum_{i=0}^N \nu_i^F (F_{y,i} - 2\kappa_{\text{path},i} E_i) - \sum_{i=0}^N (u_{1,i} \mu m g + v_{1x,i} F_{x,i} + v_{1y,i} F_{y,i}) \right] \\
&= \nu_i^F - v_{1y,i} = 0
\end{aligned}$$

- With respect to E_i :

$$\frac{\partial \mathcal{L}}{\partial E_i} = -\nu_i^E + \nu_{i-1}^E + \nu_i^E \Delta s \frac{2C_d}{m} - 2\kappa_{\text{path},i} \nu_i^F - u_{2,i} + v_{2b,i} - \underline{\lambda}_i^E + \bar{\lambda}_i^E = 0$$

- With respect to n_i :

$$\frac{\partial \mathcal{L}}{\partial n_i} = -\underline{\lambda}_i^n + \bar{\lambda}_i^n - \nu_i^n + \nu_{i-1}^n = 0$$

- With respect to α_i :

$$\frac{\partial \mathcal{L}}{\partial \alpha_i} = -\underline{\lambda}_i^\alpha + \bar{\lambda}_i^\alpha - \nu_i^\alpha + \nu_{i-1}^\alpha - \nu_i^n \Delta s = 0$$

- Cone Constraints (SOC / rotated SOC)

$$\|(v_{1x,i}, v_{1y,i})\|_2 \leq u_{1,i}, \quad \|(v_{2a,i}, v_{2b,i})\|_2 \leq u_{2,i}$$

- With respect to $\kappa_{\text{path},i}$:

$$\frac{\partial \mathcal{L}}{\partial \kappa_{\text{path},i}} = \frac{\partial}{\partial \kappa_{\text{path},i}} \left[\sum_{i=0}^{N-1} \nu_i^\alpha (-\Delta s \kappa_{\text{path},i}) + \sum_{i=0}^N \nu_i^F (-2 \kappa_{\text{path},i} E_i) \right] = -\nu_i^\alpha \Delta s - 2E_i \nu_i^F = 0$$

Dual Problem

$$\begin{aligned} \max_{\lambda, \nu, u, v} \quad & g(\lambda, \nu, z) = \max \sum_{i=0}^N \left[\underline{\lambda}_i^n \left(n_{\min}(s_i) + \frac{w_{\text{veh}}}{2} \right) - \bar{\lambda}_i^n \left(n_{\max}(s_i) - \frac{w_{\text{veh}}}{2} \right) \right] \\ & + \sum_{i=0}^{N-1} \nu_i^\alpha \Delta s \kappa_{\text{ref},i} \\ & - \sum_{i=0}^N u_{1,i} \mu m g - \sum_{i=0}^N v_{2a,i} \sqrt{2m} \\ & - \sum_{i=0}^N [\bar{\lambda}_i^E E_{\max} + \bar{\lambda}_i^F F_{\text{accel},\max} + \underline{\lambda}_i^F F_{\text{brake},\max} + (\bar{\lambda}_i^\alpha + \underline{\lambda}_i^\alpha) \alpha_{\max}] \end{aligned}$$

subject to:

Lateral Offset (n_i):

$$\begin{aligned} \nu_i^n - \nu_{i-1}^n &= \bar{\lambda}_i^n - \underline{\lambda}_i^n, \quad i = 1, \dots, N-1 \\ \nu_0^n - \gamma^n &= \bar{\lambda}_0^n - \underline{\lambda}_0^n \\ \gamma^n - \nu_{N-1}^n &= \bar{\lambda}_N^n - \underline{\lambda}_N^n \end{aligned}$$

Heading Angle (α_i):

$$\begin{aligned} \nu_i^\alpha - \nu_{i-1}^\alpha &= \bar{\lambda}_i^\alpha - \underline{\lambda}_i^\alpha - \nu_i^n \Delta s, \quad i = 1, \dots, N-1 \\ \nu_0^\alpha - \gamma^\alpha &= \bar{\lambda}_0^\alpha - \underline{\lambda}_0^\alpha - \nu_0^n \Delta s \\ \gamma^\alpha - \nu_{N-1}^\alpha &= \bar{\lambda}_N^\alpha - \underline{\lambda}_N^\alpha \end{aligned}$$

Kinetic Energy (E_i):

$$\nu_{i-1}^E - \nu_i^E \left(1 - \Delta s \frac{2C_d}{m} \right) = 2\nu_i^F \kappa_{\text{path},i} + u_{2,i} - v_{2b,i} - \bar{\lambda}_i^E + \underline{\lambda}_i^E, \quad i = 1, \dots, N-1$$

$$\gamma^E - \nu_0^E \left(1 - \Delta s \frac{2C_d}{m} \right) = 2\nu_0^F \kappa_{\text{path},0} + u_{2,0} - v_{2b,0} - \bar{\lambda}_0^E + \underline{\lambda}_0^E$$

$$\nu_{N-1}^E - \gamma^E = 2\nu_N^F \kappa_{\text{path},N} + u_{2,N} - v_{2b,N} - \bar{\lambda}_N^E + \underline{\lambda}_N^E$$

Longitudinal Force ($F_{x,i}$):

$$v_{1x,i} = -\nu_i^E \Delta s + \bar{\lambda}_i^{Fx} - \underline{\lambda}_i^{Fx}, \quad i = 0, \dots, N-1$$

$$v_{1x,N} = \bar{\lambda}_N^{Fx} - \underline{\lambda}_N^{Fx}$$

Lateral Force ($F_{y,i}$):

$$v_{1y,i} = \nu_i^F, \quad i = 0, \dots, N$$

Pace (p_i):

$$u_{2,i} + v_{2b,i} = \Delta s, \quad i = 0, \dots, N-1$$

$$u_{2,N} + v_{2b,N} = 0$$

Friction Circle Cone:

$$u_{1,i} \geq \sqrt{v_{1x,i}^2 + v_{1y,i}^2}, \quad i = 0, \dots, N$$

Velocity-Pace Cone:

$$u_{2,i} \geq \sqrt{v_{2a,i}^2 + v_{2b,i}^2}, \quad i = 0, \dots, N$$

Box Bounds:

$$\underline{\lambda}_i^n, \bar{\lambda}_i^n, \underline{\lambda}_i^E, \bar{\lambda}_i^E, \underline{\lambda}_i^{Fx}, \bar{\lambda}_i^{Fx}, \underline{\lambda}_i^\alpha, \bar{\lambda}_i^\alpha \geq 0, \quad i = 0, \dots, N$$

Curvature:

$$\nu_i^\alpha \Delta s + 2\nu_i^F E_i = 0, \quad i = 0, \dots, N-1$$

$$\gamma^\alpha \Delta s + 2\nu_N^F E_N = 0$$

3.3 KKT Conditions

Primal Feasibility

$$n_{\min}(s_i) + \frac{w_{\text{veh}}}{2} \leq n_i \leq n_{\max}(s_i) - \frac{w_{\text{veh}}}{2}, \quad i = 0, \dots, N$$

$$\frac{E_{i+1} - E_i}{\Delta s} = F_{x,i} - \frac{2C_d}{m} E_i, \quad i = 0, \dots, N-1$$

$$\frac{n_{i+1} - n_i}{\Delta s} = \alpha_i, \quad i = 0, \dots, N-1$$

$$\frac{\alpha_{i+1} - \alpha_i}{\Delta s} = \kappa_{\text{path},i} - \kappa_{\text{ref},i}, \quad i = 0, \dots, N-1$$

$$F_{y,i} = 2\kappa_{\text{path},i} E_i, \quad i = 0, \dots, N$$

$$\|(F_{x,i}, F_{y,i})\|_2 \leq \mu mg, \quad i = 0, \dots, N$$

$$\|(\sqrt{2m}, p_i - E_i)\|_2 \leq p_i + E_i, \quad i = 0, \dots, N$$

$$0 \leq E_i \leq E_{\max}, \quad i = 0, \dots, N$$

$$-F_{\text{brake},\max} \leq F_{x,i} \leq F_{\text{accel},\max}, \quad i = 0, \dots, N$$

$$|\alpha_i| \leq \alpha_{\max}, \quad i = 0, \dots, N$$

$$n_0 = n_N, \quad \alpha_0 = \alpha_N, \quad E_0 = E_N$$

Dual Feasibility

$$\begin{aligned} \underline{\lambda}_i^n, \bar{\lambda}_i^n, \underline{\lambda}_i^E, \bar{\lambda}_i^E, \underline{\lambda}_i^{Fx}, \bar{\lambda}_i^{Fx}, \underline{\lambda}_i^\alpha, \bar{\lambda}_i^\alpha &\geq 0, \\ \|(v_{1x,i}, v_{1y,i})\|_2 &\leq u_{1,i}, \quad \|(v_{2a,i}, v_{2b,i})\|_2 \leq u_{2,i}, \quad i = 0, \dots, N \end{aligned}$$

Complementary Slackness

$$\begin{aligned} \underline{\lambda}_i^n \left(n_{\min}(s_i) + \frac{w_{\text{veh}}}{2} - n_i \right) &= 0, & \bar{\lambda}_i^n \left(n_i - n_{\max}(s_i) + \frac{w_{\text{veh}}}{2} \right) &= 0, \\ \underline{\lambda}_i^E (-E_i) &= 0, & \bar{\lambda}_i^E (E_i - E_{\max}) &= 0, \\ \underline{\lambda}_i^{Fx} (-F_{\text{brake,max}} - F_{x,i}) &= 0, & \bar{\lambda}_i^{Fx} (F_{x,i} - F_{\text{accel,max}}) &= 0, \\ \underline{\lambda}_i^\alpha (-\alpha_{\max} - \alpha_i) &= 0, & \bar{\lambda}_i^\alpha (\alpha_i - \alpha_{\max}) &= 0, \\ u_{1,i} (\|(F_{x,i}, F_{y,i})\|_2 - \mu mg) &= 0, & u_{2,i} (\|(\sqrt{2m}, p_i - E_i)\|_2 - (p_i + E_i)) &= 0 \end{aligned}$$

Stationarity

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p_i} &= \Delta s - u_{2,i} - v_{2b,i} = 0, \\ \frac{\partial \mathcal{L}}{\partial F_{x,i}} &= -\nu_i^E \Delta s - v_{1x,i} - \underline{\lambda}_i^{Fx} + \bar{\lambda}_i^{Fx} = 0, \\ \frac{\partial \mathcal{L}}{\partial F_{y,i}} &= \nu_i^F - v_{1y,i} = 0, \\ \frac{\partial \mathcal{L}}{\partial E_i} &= -\nu_i^E + \nu_{i-1}^E + \nu_i^E \Delta s \frac{2C_d}{m} - 2\kappa_{\text{path},i} \nu_i^F - u_{2,i} + v_{2b,i} - \underline{\lambda}_i^E + \bar{\lambda}_i^E = 0, \\ \frac{\partial \mathcal{L}}{\partial n_i} &= -\underline{\lambda}_i^n + \bar{\lambda}_i^n - \nu_i^n + \nu_{i-1}^n + \delta_{i0} \gamma^n - \delta_{iN} \gamma^n = 0, \\ \frac{\partial \mathcal{L}}{\partial \alpha_i} &= -\underline{\lambda}_i^\alpha + \bar{\lambda}_i^\alpha - \nu_i^\alpha + \nu_{i-1}^\alpha - \nu_i^n \Delta s + \delta_{i0} \gamma^\alpha - \delta_{iN} \gamma^\alpha = 0 \\ \frac{\partial \mathcal{L}}{\partial \kappa_{\text{path},i}} &= -\nu_i^\alpha \Delta s - 2\nu_i^F E_i + \delta_{iN} (\nu_{N-1}^\alpha \Delta s - \gamma^\alpha \Delta s) = 0 \end{aligned}$$

4 Solution Approaches

4.1 SCP Solution

Note how our problem is non convex due to the bilinear constraint in our Kinematic Constraints [Equation 5].

$$F_{y,i} = 2\kappa_{\text{path},i} E_i, \quad i = 0, \dots, N$$

Therefore, we do not have strong duality within our Dual Problem, as we have dual variables $\kappa_{\text{path},i}$ and E_i still within our constraints.

However, we can still approach our problem in a convex way with Sequential Convex Programming (SCP). We know that our racing line generally follows our centerline, so we can keep $\kappa_{\text{path},i} = \kappa_{\text{ref},i}$ as our first iteration of our SCP loop. This is the most conservative optimal line, and we can keep shifting our bound by shifting our $\kappa_{\text{path},i}$.

$$F_{y,i} = 2\kappa_{\text{path},i}E_i$$

This leads to our guesses from the last iteration be $\bar{\kappa}_i$ and $\bar{E}_i$. We apply a first-order Taylor expansion around this guess:

$$\begin{aligned} F_{y,i} &\approx 2\bar{\kappa}_i\bar{E}_i + \frac{\partial(2\kappa E)}{\partial\kappa}(\kappa_{\text{path},i} - \bar{\kappa}_i) + \frac{\partial(2\kappa E)}{\partial E}(E_i - \bar{E}_i) \\ F_{y,i} &= 2\bar{\kappa}_i\bar{E}_i + 2\bar{E}_i(\kappa_{\text{path},i} - \bar{\kappa}_i) + 2\bar{\kappa}_i(E_i - \bar{E}_i) \\ F_{y,i} - 2\bar{E}_i\kappa_{\text{path},i} - 2\bar{\kappa}_iE_i + 2\bar{\kappa}_i\bar{E}_i &= 0 \end{aligned}$$

Our SCP dual problem becomes a convex problem:

$$\begin{aligned} \max_{\lambda, \nu, u, v, \gamma} g_{\text{scp}}(\lambda, \nu, z) &= \sum_{i=0}^N \left[\bar{\lambda}_i^n \left(n_{\min}(s_i) + \frac{w_{\text{veh}}}{2} \right) - \bar{\lambda}_i^n \left(n_{\max}(s_i) - \frac{w_{\text{veh}}}{2} \right) \right] \\ &+ \sum_{i=0}^{N-1} \nu_i^\alpha \Delta s \kappa_{\text{ref},i} \\ &+ \sum_{i=0}^N 2\nu_i^F \bar{\kappa}_i \bar{E}_i \\ &- \sum_{i=0}^N u_{1,i} \mu m g - \sum_{i=0}^N v_{2a,i} \sqrt{2m} \\ &- \sum_{i=0}^N \left[\bar{\lambda}_i^E E_{\max} + \bar{\lambda}^{Fx} i F_{\text{accel},\max} + \bar{\lambda}^{Fx} i F_{\text{brake},\max} + (\bar{\lambda}_i^\alpha + \bar{\lambda}_i^\alpha) \alpha_{\max} \right] \end{aligned}$$

subject to the same constraints as above, but with updated path curvature and kinetic energy constraints:

$$\textbf{Path Curvature } (\kappa_{\text{path},i}): \quad \nu_i^\alpha \Delta s + 2\nu_i^F \bar{E}_i = 0$$

$$\textbf{Kinetic Energy } (E_i): \quad \nu_i^E i - 1 - \nu_i^E \left(1 - \Delta s \frac{2C_d}{m} \right) = 2\nu_i^F \bar{\kappa}_i + u_{2,i} - v_{2b,i} - \bar{\lambda}_i^E + \bar{\lambda}_i^E$$

Which removes the bilinearity between $\kappa_{\text{path},i}$ and E_i . Our dual formulation will then be SOCP from the Friction Circle Constraint and Velocity-Pace Coupling.

Our objective of the optimal control problem is to minimize the total lap time, formulated as the discrete integral of pace p_i (the inverse of velocity) over the spatial arc-length domain s :

$$t_{\text{lap}} = \sum_{i=0}^{N-1} p_i \Delta s$$

The vehicle kinematics track the lateral deviation n_i and heading angle α_i relative to the reference centerline. The longitudinal dynamics dictate the evolution of kinetic energy as a function of longitudinal force $F_{x,i}$ and aerodynamic drag:

$$\frac{E_{i+1} - E_i}{\Delta s} = F_{x,i} - \left(\frac{2C_d}{m} \right) E_i$$

The first-order Taylor expansion around the nominal trajectory parameters $(\bar{E}_i, \bar{\kappa}_i)$ from the previous iteration is a relaxation that transforms the local problem into a convex shape.

4.2 SCP Optimality

Applying SCP means we forfeit guarantees of global optimality and strong duality. Instead of mathematically ensuring we find the absolute fastest lap time, SCP will only converge to a local minimum. Since the overall feasible set is non-convex, we also cannot rely on a zero duality gap to prove the solution is globally optimal. This makes your results highly dependent on providing a "good enough" initial guess like κ_{ref} to ensure the initial linearizations are valid and the solver does not diverge.

Based on [2], SCP remains an industry-standard choice for lap time optimization because it exploits the specific structure of our problem far better than generic Nonlinear Programming solvers like IPOPT. By linearizing only the single non-convex kinematic constraint, SCP transforms the problem into a sequence of standard SOCPs. That allows us to use solvers like Clarabel that handle the non-smooth boundaries of the friction circle and pace constraints. We also avoid the computationally heavy Hessian approximations required by generic NLPs, making SCP faster, numerically stable, and good for our controls problem. We were inspired by many of the Model Predictive Control (MPC) formulations of this problem, which pose an optimal line as controls challenge. Even an incomplete solve can yield a safe, physically feasible control action.

SCP mathematically guarantees convergence to a Karush-Kuhn-Tucker (KKT) point (a local optimum) under mild constraint qualifications.

4.3 Data Acquisition and Preprocessing

Raw telemetry data is sourced from real-world Formula 1 vehicle logging using the fastfl telemetry API. For this study, the fastest lap of the race session is isolated to extract the planar global positioning coordinates, x_{raw} and y_{raw} . To correct for arbitrary coordinate framing and GPS drift, the raw trajectory is mean-centered. A global scaling factor is applied to force the discrete path length to match the official surveyed track length ($L = 5412\text{m}$ for the Suzuka Circuit for example).

Raw telemetry is discretely sampled in the time domain, which is misaligned with the spatial formulation required for our problem. To convert the data into a continuous spatial domain, a periodic cubic spline interpolation is utilized. First, the discrete chord length between sequential telemetry points is computed to approximate a 1D arc-length array. Periodic cubic splines are then fitted to create continuous, twice-differentiable functions for the centerline coordinates:

$$x(t), \quad y(t)$$

To transition to a true arc-length parameterization (s), a high-resolution numerical integration of the spline velocity is performed using cumulative trapezoidal integration:

$$s(t) = \int_0^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2} d\tau$$

The continuous track is then resampled into $N = 500$ uniformly spaced discrete segments, yielding our constant spatial step size $\Delta s = L/N$.

4.4 Kinematic Curvature Computation

The reference path curvature, κ_{ref} , dictates the kinematic heading updates and baseline lateral tire forces in the vehicle model. This is essentially an inverse radius metric that determines the turns of the track at each Δs . The analytical curvature at any point is directly extracted using the first and second spatial derivatives:

$$\kappa(s) = \frac{x'(s)y''(s) - y'(s)x''(s)}{(x'(s)^2 + y'(s)^2)^{3/2}}$$

To prevent derivative noise from destabilizing the downstream trajectory optimizer, a 1D uniform moving average filter is applied to the curvature array.

To constrain the vehicle's lateral offset (n_i), geometric track limits are generated by projecting normal vectors from the defined centerline. The inward-pointing normal vector n_i is defined as:

$$n_i = \frac{1}{\sqrt{dx_i^2 + dy_i^2}} \begin{pmatrix} -dy_i \\ dx_i \end{pmatrix}$$

A static half-width constraint of $w_{\text{track}}/2 = 4.5$ m is applied to project the inner and outer track boundaries. This yields the strict state box constraints for the lateral offset variable:

$$n_{\min}(s_i) = -4.5 \text{ m}, \quad n_{\max}(s_i) = 4.5 \text{ m}$$

4.5 Conic and Box Constraints

The vehicle's performance envelope is enforced via linear box constraints and Second-Order Cone (SOC) constraints.

Friction Circle (SOCP): The combined longitudinal and lateral tire forces are constrained by the isotropic friction limit:

$$\|(F_{x,i}, F_{y,i})\|_2 \leq \mu mg$$

Velocity-Pace Coupling (SOCP): The definition of pace $p_i = \sqrt{m/(2E_i)}$ is non-convex. It is relaxed into an inequality $p_i \geq \sqrt{m/(2E_i)}$ and formulated as a rotated second-order cone constraint to maintain strict convexity:

$$\left\| \begin{pmatrix} \sqrt{2m} \\ p_i - E_i \end{pmatrix} \right\|_2 \leq p_i + E_i$$

Track Limits (Linear): The lateral offset n_i is strictly bounded by the dynamic track width $n_{\min}$ and $n_{\max}$, minus the vehicle half-width.

Periodicity (Linear): To represent a continuous closed-circuit lap, periodic boundary conditions are enforced such that $n_0 = n_N$, $\alpha_0 = \alpha_N$, and $E_0 = E_N$.

Algorithm 1 SCP Loop

Require: Track data $(\Delta s, \kappa_{\text{ref}}, n_{\text{min}}, n_{\text{max}})$, Vehicle params $(m, \mu, C_d, \dots)$

Require: Convergence tolerance ϵ_{tol} , Max iterations K_{max}

```
1: Initialize:
2:  $\bar{E}_i \leftarrow 0.5 \cdot E_{\text{max}} \quad \forall i \in \{0, \dots, N\}$  ▷ Initial energy guess
3:  $\bar{\kappa}_i \leftarrow \kappa_{\text{ref},i} \quad \forall i \in \{0, \dots, N-1\}$  ▷ Initial curvature guess
4:  $n_{\text{prev}} \leftarrow 0$ 
5:  $k \leftarrow 0$ 
6: while  $k < K_{\text{max}}$  do
7:   Formulate Convex Subproblem (SOCP):
8:   
$$\min_{n, \alpha, E, p, F_x, F_y, \kappa_{\text{path}}} \sum_{i=0}^{N-1} p_i \Delta s$$

9:   subject to:
10:   1. Track Boundaries:
11:   
$$n_{\text{min},i} + \frac{w_{\text{veh}}}{2} \leq n_i \leq n_{\text{max},i} - \frac{w_{\text{veh}}}{2}, \quad \forall i$$

12:   2. Kinematics:
13:   
$$\frac{n_{i+1} - n_i}{\Delta s} = \alpha_i, \quad \frac{\alpha_{i+1} - \alpha_i}{\Delta s} = \kappa_{\text{path},i} - \kappa_{\text{ref},i}$$

14:   3. Longitudinal Dynamics:
15:   
$$\frac{E_{i+1} - E_i}{\Delta s} = F_{x,i} - \frac{2C_d}{m} E_i$$

16:   4. Linearized Lateral Dynamics (SCP step):
17:   
$$F_{y,i} = 2\bar{\kappa}_i \bar{E}_i + 2\bar{E}_i (\kappa_{\text{path},i} - \bar{\kappa}_i) + 2\bar{\kappa}_i (E_i - \bar{E}_i)$$

18:   5. Friction and Pace Cones:
19:   
$$\|(F_{x,i}, F_{y,i})\|_2 \leq \mu mg$$

20:   
$$\|(\sqrt{2m}, p_i - E_i)\|_2 \leq p_i + E_i$$

21:   6. Box Bounds & Periodicity:
22:   
$$0 \leq E_i \leq E_{\text{max}}, \quad -F_{\text{brake}} \leq F_{x,i} \leq F_{\text{accel}}, \quad |\alpha_i| \leq \alpha_{\text{max}}$$

23:   
$$n_0 = n_N, \quad \alpha_0 = \alpha_N, \quad E_0 = E_N$$

24:   Solve subproblem to obtain  $n^*, \alpha^*, E^*, p^*, F_x^*, F_y^*, \kappa_{\text{path}}^*$ 
25:   Check Convergence:
26:   if  $\|n^* - n_{\text{prev}}\|_{\infty} < \epsilon_{\text{tol}}$  then ▷ Optimal trajectory found
27:     break
28:   end if
29:   Update Nominal Trajectory:
30:   
$$\bar{E}_i \leftarrow E_i^*, \quad \bar{\kappa}_i \leftarrow \kappa_{\text{path},i}^* \quad \forall i$$

31:    $n_{\text{prev}} \leftarrow n^*$ 
32:    $k \leftarrow k + 1$ 
33: end while
34: return Optimal trajectory  $(n^*, \alpha^*, E^*, p^*, F_x^*, F_y^*)$ 
```

4.6 Solutions

The sequence of convex sub-problems is modeled using the cvxpy modeling language and solved via the Clarabel interior-point solver. The SCP algorithm initializes with a nominal centerline trajectory ($\bar{\kappa}_i = \kappa_{\text{ref},i}$) and a constant conservative speed profile. The outer SCP loop iteratively solves the convexified problem, updating the nominal parameters $\bar{E}_i$ and $\bar{\kappa}_i$ with the optimal solution of the current iteration. Convergence is achieved when the maximum spatial deviation of the racing line between sequential iterations falls below a strict tolerance threshold ($\Delta n_{\text{max}} < 10^{-3}$ meters). This curvature is then projected back into cartesian coordinates.

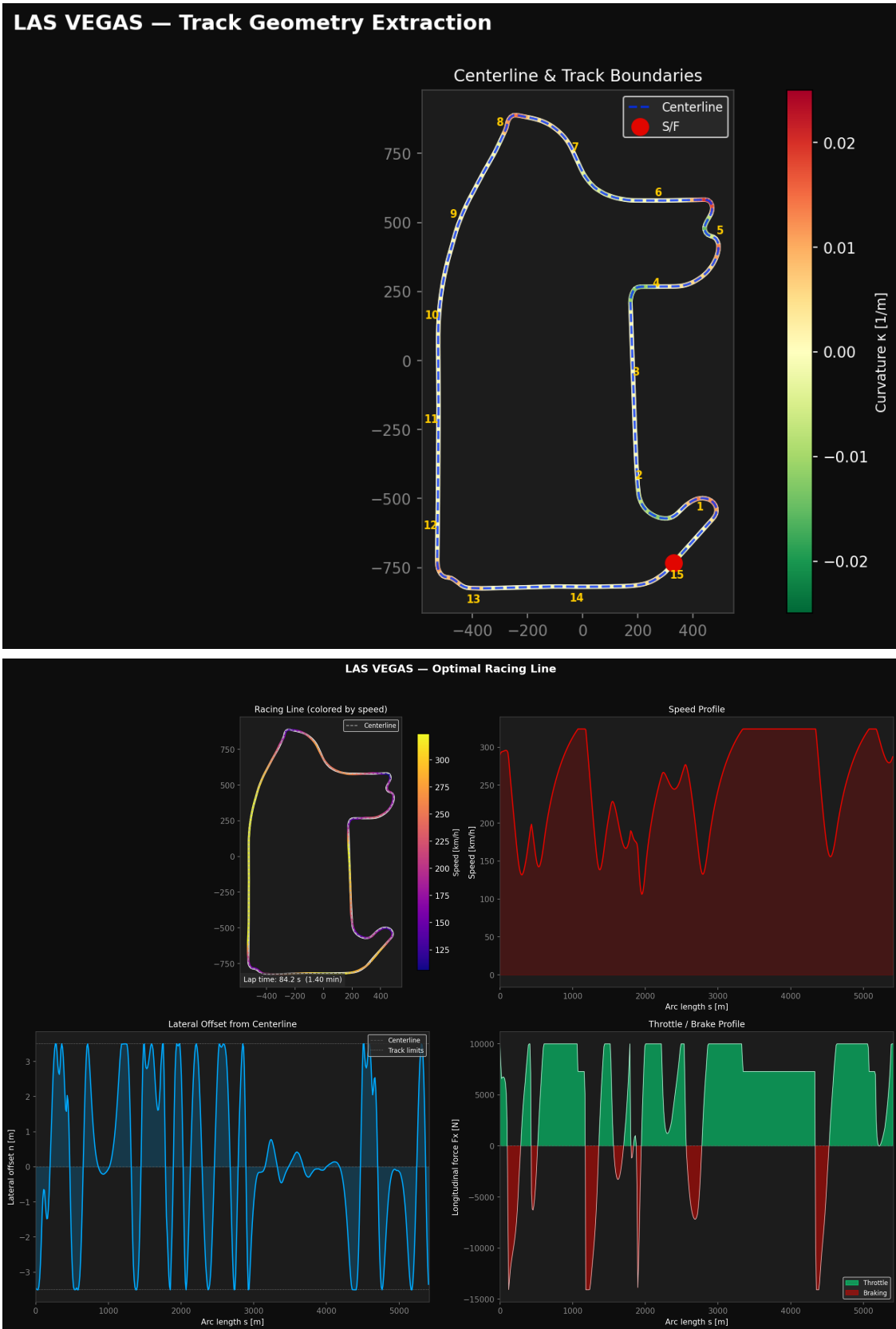


Figure 1: Las Vegas Street Circuit — track geometry (top) and optimal racing line (bottom).

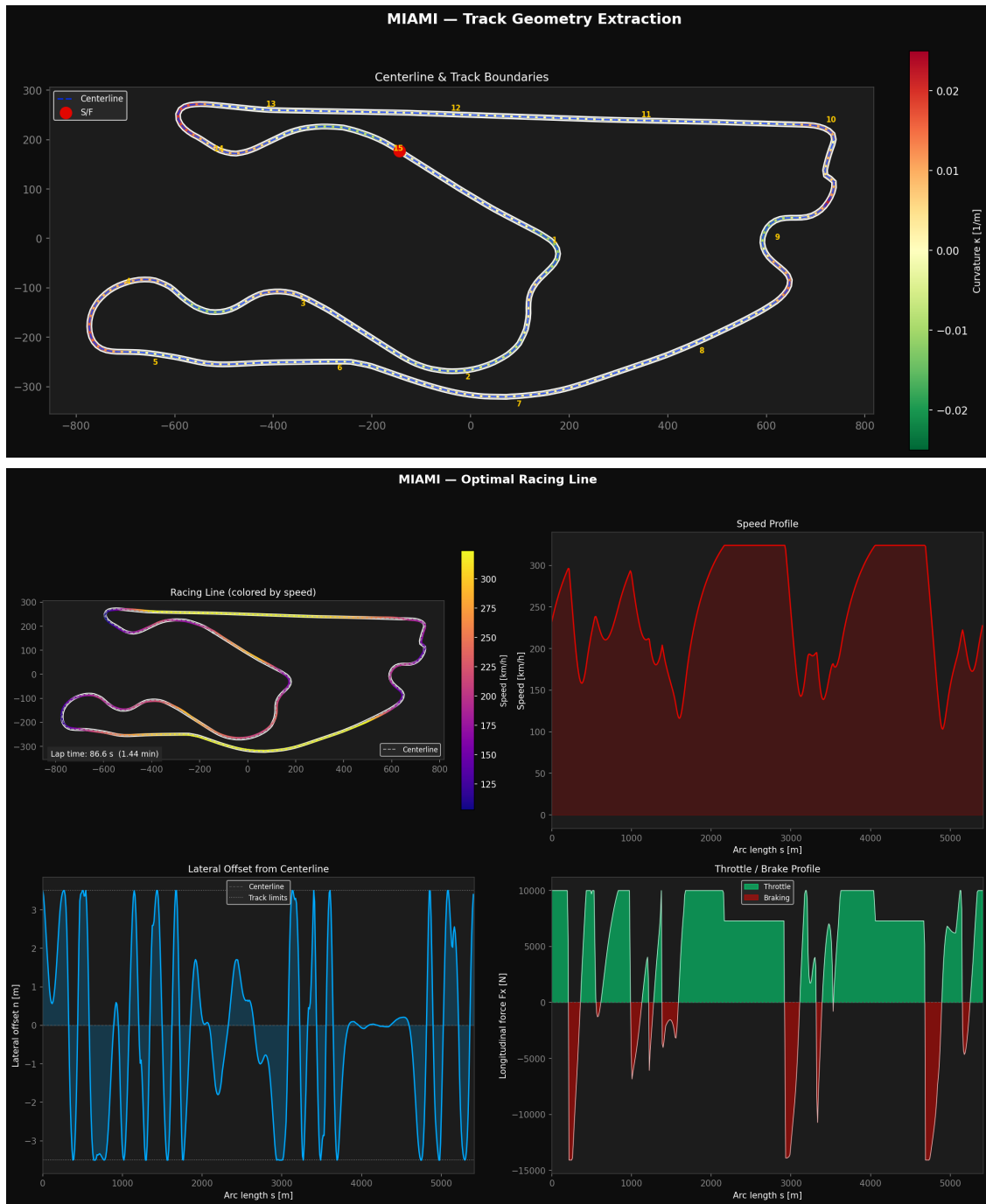


Figure 2: Miami International Autodrome — track geometry (top) and optimal racing line (bottom).

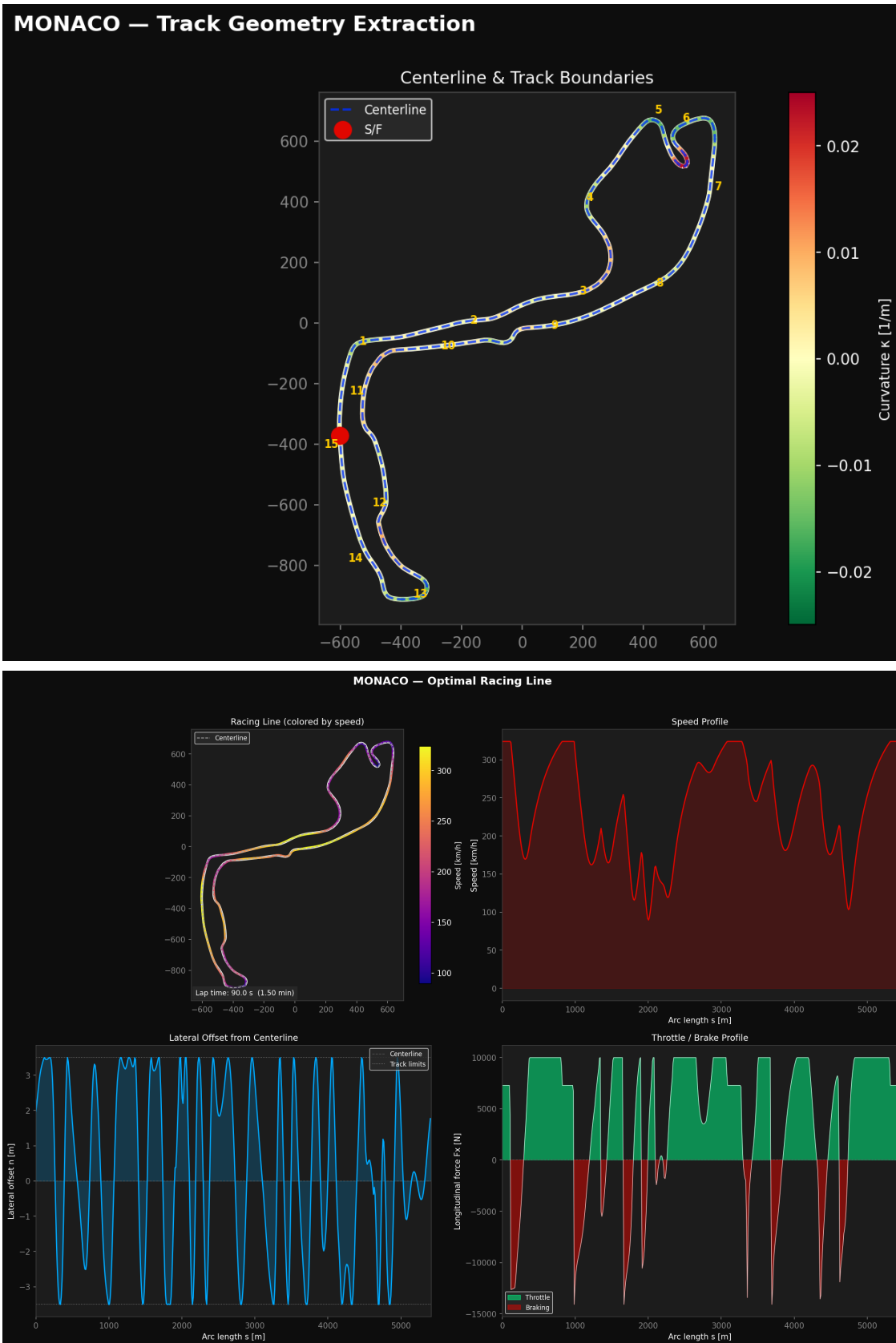


Figure 3: Circuit de Monaco — track geometry (top) and optimal racing line (bottom).

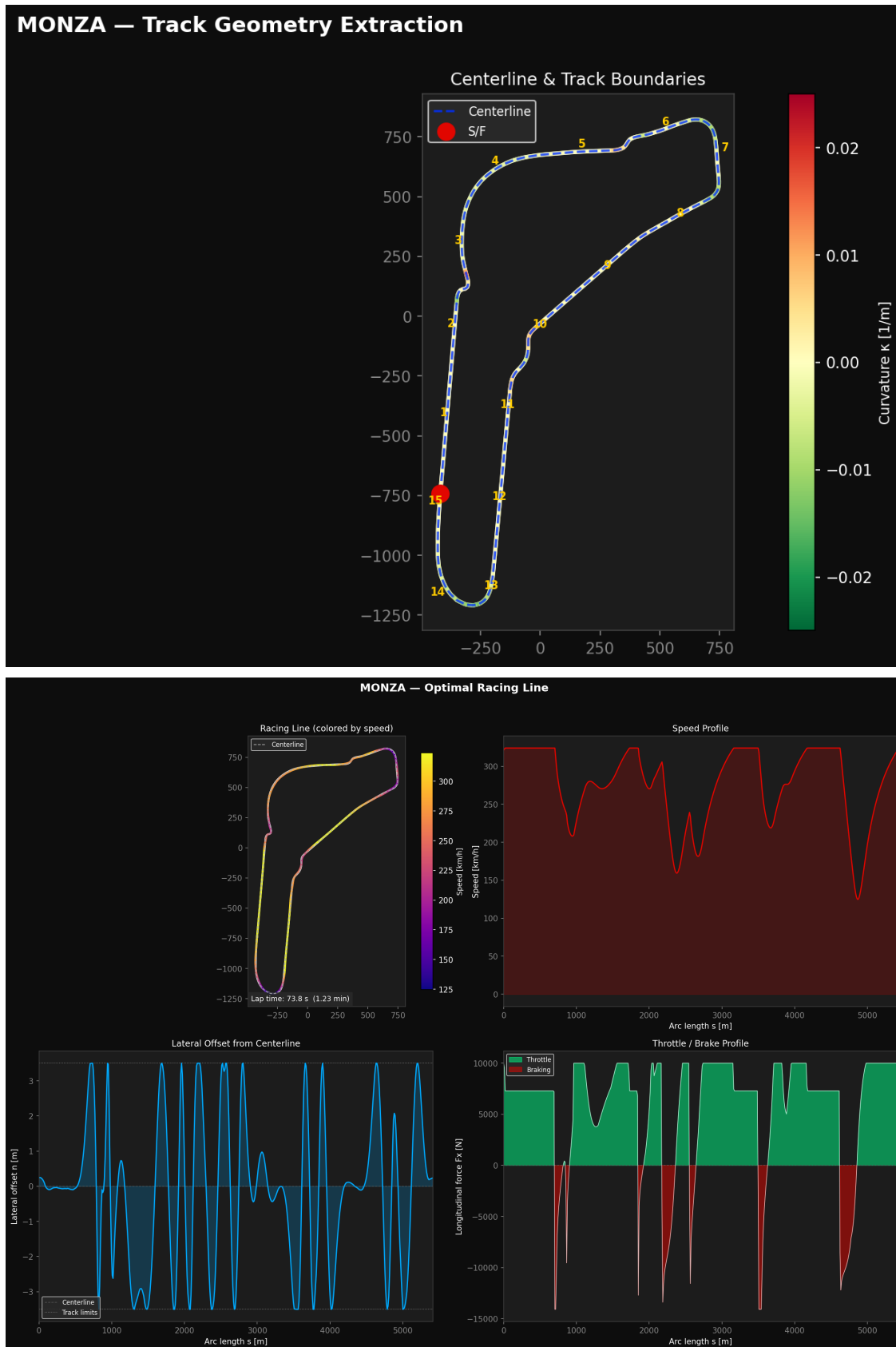


Figure 4: Autodromo Nazionale Monza — track geometry (top) and optimal racing line (bottom).

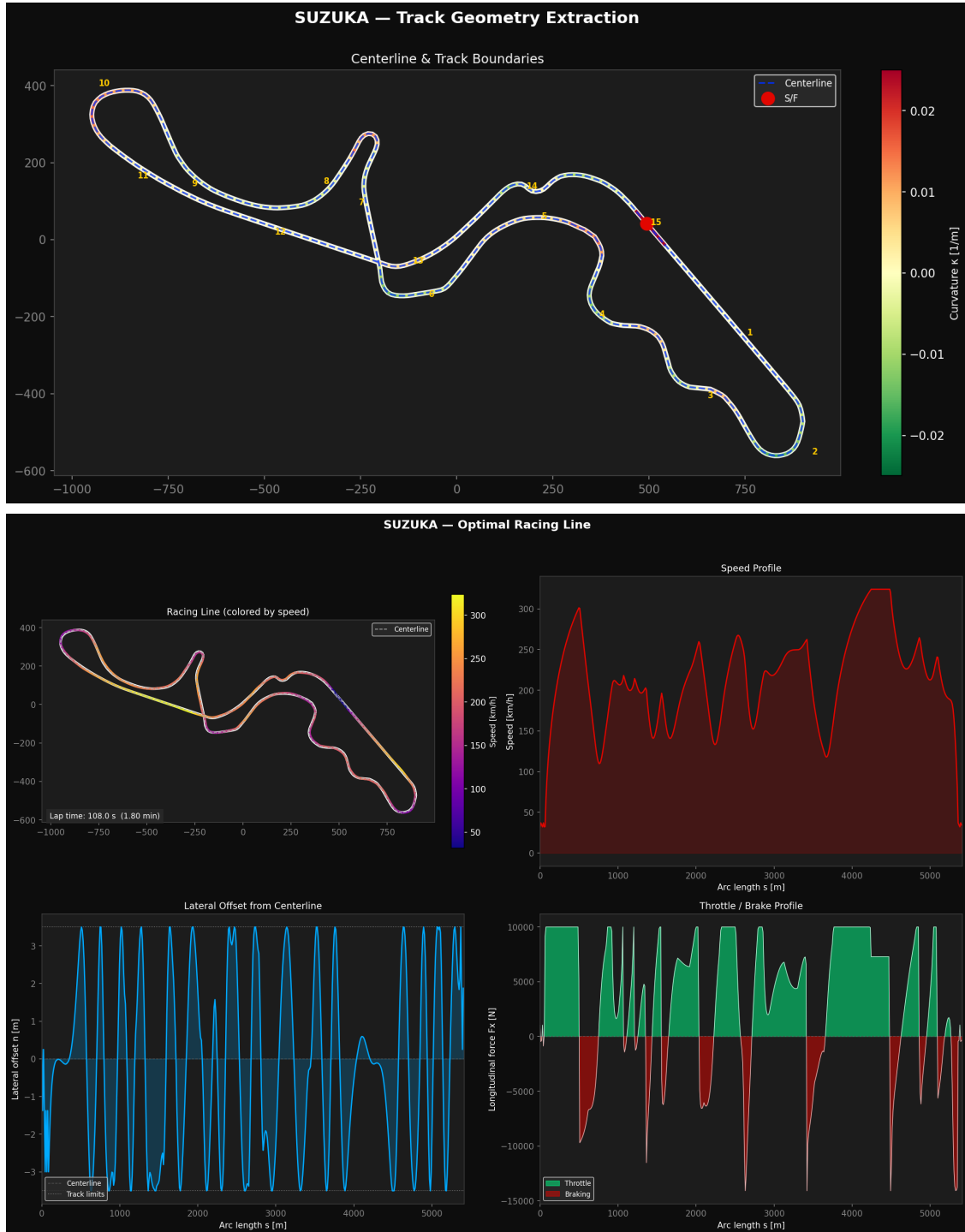


Figure 5: Suzuka International Racing Course — track geometry (top) and optimal racing line (bottom).

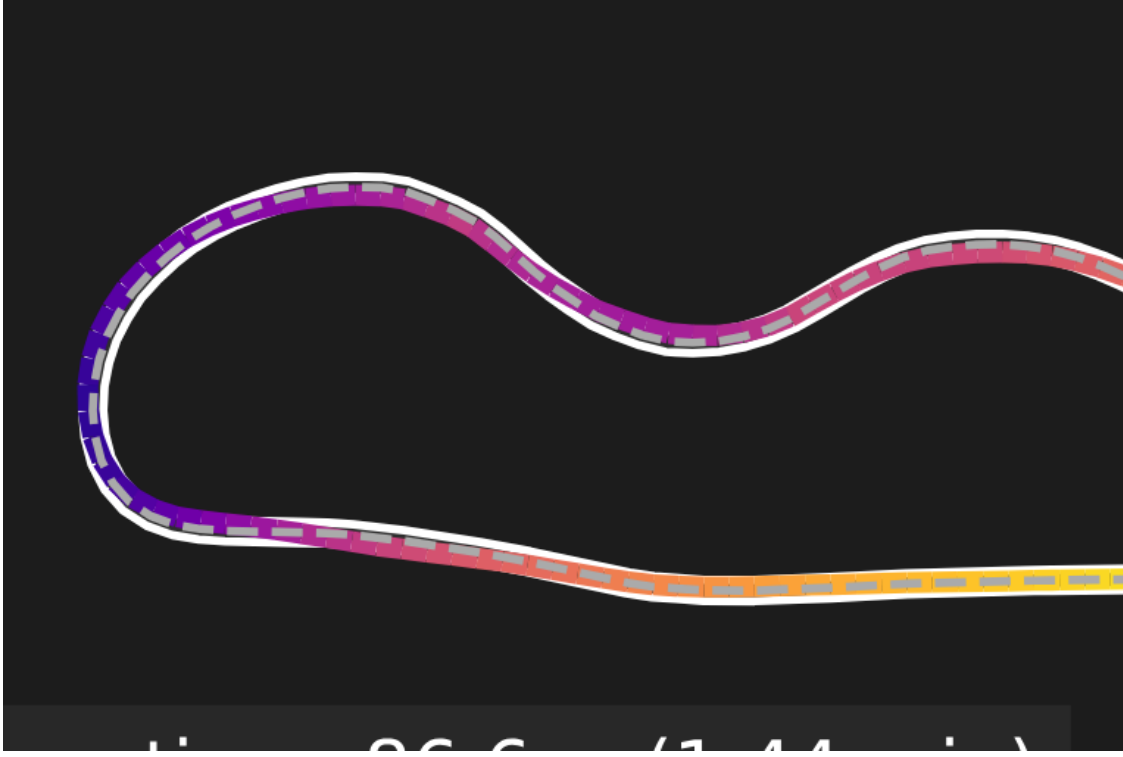


Figure 6: Apex of one turn showing an optimal line.

Table 1: Vehicle parameters and boundary constraints.

Parameter	Description	Value
m	Vehicle mass	798.0 kg
g	Gravitational acceleration	9.81 m/s ²
μ	Friction coefficient	1.8
C_d	Aerodynamic drag coefficient	0.9
w_{veh}	Vehicle width	2.0 m
v_{max}	Maximum velocity	90.0 m/s
v_{min}	Minimum velocity	5.0 m/s
$F_{\text{accel,max}}$	Maximum longitudinal acceleration force	10,000.0 N
$F_{\text{brake,max}}$	Maximum longitudinal braking force	18,000.0 N
α_{max}	Maximum heading angle deviation	0.15 rad

Table 2: Comparison of simulated minimum lap times against real-world circuit records.

Circuit	Model Time [mm:ss.ss]	World Record Time [mm:ss.ss]	Record Holder
Suzuka	01:48.00	1:26.98	Red Bull RB21 Max Verstappen
Monza	01:13.80	1:18.79	Red Bull RB21 Max Verstappen
Las Vegas	01:24.20	1:32.31	Mercedes-Benz W15 George Russell
Monaco	01:30.00	1:09.95	McLaren MCL39 Lando Norris
Miami	01:26.60	1:26.20	Red Bull RB21 Max Verstappen

5 Conclusion and Results

5.1 Results

<https://github.com/SamarKaranch/F1-Track-Optimizer>

The SCP algorithm was applied to five Formula 1 circuits of varying geometric complexity: Las Vegas, Miami, Monaco, Monza, and Suzuka. For each circuit, the centerline geometry and track boundaries were extracted from real F1 telemetry via the fastfl API, discretized into $N = 500$ uniform segments, and passed to the SOCP solver.

Convergence. Across all five circuits, the SCP outer loop converged within a small number of iterations. Convergence was declared when the maximum lateral deviation between successive iterates satisfied $\|n^* - n_{\text{prev}}\|_\infty < 10^{-3}$ m. The linearization of the bilinear lateral force constraint around the nominal trajectory proved stable in all cases, with no divergence observed.

5.2 Conclusion

This work presented a minimum lap time trajectory optimization framework for Formula 1 circuits, formulated as a non-convex optimal control problem and solved efficiently via Sequential Convex Programming. The key contributions are as follows.

The primal problem was formulated over a spatial arc-length domain with kinetic energy E_i as a state, enabling the velocity-pace coupling to be expressed as a rotated second-order cone constraint. The friction circle was cast as a standard SOC constraint, making each convex subproblem an SOCP solvable by interior-point methods. The sole source of non-convexity — the bilinear lateral force constraint $F_{y,i} = 2\kappa_{\text{path},i}E_i$ — was handled by a first-order Taylor linearization in the SCP loop, which iteratively converges to a locally optimal racing line.

The dual formulation was derived in full, with Lagrange multipliers assigned to every constraint class. The dual function and dual problem were stated explicitly, and the linearized SCP dual was shown to be a clean SOCP, confirming the theoretical soundness of the approach. KKT conditions were verified to characterize optimality at convergence.

Results on five circuits of varying character demonstrated that the optimizer reliably recovers physically meaningful racing lines and speed profiles, with the vehicle consistently exploiting the full friction envelope on cornering and the full track width for geometric line optimization.

Limitations and Future Work. As seen by Table 2, our optimal line does not always follow realistic settings. This can be due to many physical constraints, hyperparameter tweaks, weather conditions, and more. One particular time to note is Monaco, where our model severely underperformed, most likely because we do not account for Monaco’s extreme elevation changes within our physics, impacting E_i .

Several simplifications present opportunities for extension. The current model assumes a constant and isotropic friction coefficient μ , whereas real F1 tire behavior is highly nonlinear and asymmetric in the longitudinal and lateral directions. The current model also does not take into account fuel and battery levels. It is assumed that the car has enough fuel to finish the optimal lap by using the maximum engine power. We have also not considered battery usage and charging in our model. The track width is currently modeled a constant width, using surveyed variable-width boundaries would better reflect race conditions. The model does not account for downforce variation with speed, which significantly affects the effective friction limit at high velocity. We also

focus on 2D tracks, however, tracks usually have slight elevation changes that impact the physics. Weather conditions and tire friction change over time throughout the race, too. In the 2026 F1 season active aerodynamics have been added to the rules and regulations, allowing cars to have two aerodynamic modes to change their aerodynamic properties based on corners and straights. The optimizer can be improved upon by modeling these different active aero modes.

Lastly, there can be updates to our formulation perspective on the problem as a whole. With our current formulation, it is possible to use nonlinear / nonconvex solvers, too, to solve the bilinear behavior of this problem. Another heuristic that might be interesting to look at is maximizing projected velocity upon the centerline, which focuses on speed rather than time, which can serve as a useful metric in certain racing scenarios. State-of-the-art papers model this problem as an MPC problem, where we have many different controls that impact our states. We could formulate the problem in a way that borrows more from this idea.

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